HAMILTON-JACOBI-BELLMAN APPROACH FOR THE CLIMBING PROBLEM FOR MULTI-STAGE LAUNCHERS*

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Abstract. In this paper we investigate the Hamilton-Jacobi-Bellman (HJB) approach for solving a complex real-world optimal control problem in high dimension. We consider the climbing problem for the European launcher Ariane V: The launcher has to reach the Geostationary Transfer Orbit with minimal propellant consumption under state/control constraints. In order to circumvent the well-known curse of dimensionality, we reduce the number of variables in the model exploiting the specific features concerning the dynamics of the mass. This generates a non–standard optimal control problem formulation. We show that the joint employment of the most advanced mathematical techniques for the numerical solution of HJB equations allows one to achieve practicable results in reasonable time.

Key words. Dynamic programming principle, minimum time problem, Ultra Bee scheme, curse of dimensionality

AMS subject classifications. 49L20, 93C15

1. Introduction.

In this paper we investigate the Hamilton-Jacobi-Bellman (HJB) approach for solving a complex real-world optimal control problem in high dimension. We aim at showing that the joint employment of the most advanced mathematical techniques for the numerical solution of HJB equations allows one to achieve practicable results in reasonable time.

The HJB approach is based on the Dynamic Programming Principle (DPP) and it can be used to solve fully nonlinear control problems and minimum time problems, taking into account mixed state/control constraints [2]. Once the HJB equation is solved, it is possible to recover the optimal control in feedback form starting form any initial condition of the state of the system. Moreover, any knowledge of the solution is needed in advance (presence of singular arcs, number of commutations and so on) and the reconstructed optimal control is the *global* minimum of the cost functional associated to the problem. All these features make this approach preferable to other more common techniques as direct methods or Pontryagin-based optimization algorithms such as shooting methods [3]. On the other hand, the HJB appraoch suffers from the so-called *curse of dimensionality*, meaning that the CPU time needed to compute a numerical solution of the HJB equation grows exponentially with respect to the state dimension, and it is currently prohibitive if the dimension is > 4. This is the main reason which makes the HJB approach an impracticable choice by engineers [10].

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In this paper we combine several recent techniques for the HJB approach, thus obtaining relatively accurate solutions to a fully nonlinear control problem in reasonable time without parallel computation. More precisely, we investigate the potential of the HJB approach for the climbing problem in the case of the European launcher Ariane V developed by the French space agency "Centre National d'Études Spatiales" (CNES). For a given payload (fixed final mass), we aim at steering the launcher to the Geostationary Transfer Orbit (GTO) with *minimal* propellant consumption under a dynamic pressure constraint [8, 11]. In this study, since the engine is always at full thrust, minimizing the propellent consumption corresponds to reach the GTO orbit in minimal time.

2. The climbing problem for multi-stage launchers. The launcher Ariane V is equipped with 2 boosters, 1 main stage and 1 secondary stage. These reservoirs are initially filled with propellant and are progressively released during flight. The flight is divided in four phases, summarized in the following table.

phase	*thrust*	*notes*
0	boosters and main stage	dynamics not controlled
1	boosters and main stage	lasts until booster separation
2	main stage	lasts until main stage separation
3	secondary stage	lasts until secondary stage separation

The complete dynamics can be described by 6 variables, in addition to time, launcher's mass (not constant), and control variable. See [11] and references therein for details. Under the assumptions that the plane of motion is in the equatorial plane and the drag and thrust forces are contained in this plane, the launcher's dynamics can be simplified as

$$\left\{\begin{array}{rcl} \dot r &=& v\cos\gamma \\ \dot v &=& -\frac{\mu}{r^2}\cos\gamma - \frac{F_D(r,v,\alpha)}{m(t)} + \frac{F_T(r)}{m(t)}\cos\alpha + \Omega^2 r\cos\gamma \\ \dot\gamma &=& \sin\gamma\left(\frac{\mu}{r^2 v} - \frac{v}{r}\right) - \frac{F_T(r)}{v\,m(t)}\sin\alpha - \Omega^2\frac{r}{v}\sin\gamma - 2\Omega \end{array}\right. \tag{2.1}$$

where r is the altitude, v is the modulus of the velocity, γ is the path inclination, α (*control variable*) is the angle of attack, m is the mass, μ is the Earth's gravitational constant, F_D is the drag force, F_T is the thrust force and Ω is the Earth's angular velocity. We also take into account a mixed state/control constraint of the form $Q(r,v)|\alpha| \le C_s$, where C_s is a constant and Q is the dynamic pressure. The dynamic pressure takes large values into the atmosphere, and this forces the angle of attack to be very small.

The mass $m = m(t)$ varies during the flight. It is given by the sum of three quantities: payload's mass M_{PL}, launcher's mass M_L and propellant's mass M_P. M_P decreases continuously as the fuel is consumed, the rate of consumption depends on the phase and it is constant in each phase. At time $t = 0$ the launcher's mass M_L is given by the sum of the 2 stages and the boosters and then drops abruptly at the end of each phase, since reservoirs are progressively released.

Let us remark that while $(\dot r, \dot v, \dot\gamma)$ depend on m, $\dot m$ does not depend on the other variables and on the control variable. This important property allows us not to include the mass in the state variables of the HJB equation. Indeed, assuming the final time t_f of the trajectory is known, the mass function $m(t)$ can be computed by means of a simple backward integration with the "initial" condition $m(t_f) = M_{PL}$.

Let us define a new function $\widetilde{m}$ as the function such that $\widetilde{m}(t_f - t) = m(t)$ for any $t \in [0, t_f]$. We can re-write the dynamics of the system as follows

$$\left\{\begin{array}{lcl} \dot{r} & = & v\cos\gamma \\ \dot{v} & = & -\frac{\mu}{r^2}\cos\gamma - \frac{F_D(r,v,\alpha)}{\widetilde{m}(t_f-t)} + \frac{F_T(r)}{\widetilde{m}(t_f-t)}\cos\alpha + \Omega^2 r\cos\gamma \\ \dot{\gamma} & = & \sin\gamma\left(\frac{\mu}{r^2 v} - \frac{v}{r}\right) - \frac{F_T(r)}{v\,\widetilde{m}(t_f-t)}\sin\alpha - \Omega^2\frac{r}{v}\sin\gamma - 2\Omega. \end{array}\right. \tag{2.2}$$

In this equivalent formulation, the mass does not appear as a state variable but it is explicitly given as a function of the final time. Because of the one-to-one relation between the time t and the mass $m(t)$, minimizing the propellant consumption is equivalent to minimizing the time t_f used to reach the target.

We therefore consider a minimum time problem of the form

$$(\mathcal{P})\qquad \left\{\begin{array}{l} \mathcal{T}(x) := \text{minimize } \ t_f, \\ \\ \text{with } \left\{\begin{array}{l} \dot{y}(t) = f(t_f - t, y(t), \alpha(t))\,,\ t \in [0, t_f], \\ y(0) = x, \end{array}\right. \\ \qquad t_f \geq 0, \qquad \alpha(t) \in \mathcal{A} \quad \text{for a.e. } t \in [0, t_f], \\ \qquad y(t_f) \in \mathcal{C}, \qquad \Psi(y(t), \alpha(t)) \leq 0 \ \text{ for a.e. } t \in [0, t_f], \\ \qquad g(y(t)) \leq 0 \text{ for all } t \in [0, t_f], \end{array}\right.$$

where $y = (r, v, \gamma)$ is the state of the system, its initial condition x belongs to $\mathbb{R}^d$ (with $d = 3$), $\mathcal{A} \subset \mathbb{R}^m$ (with $m = 1$) is the set of admissible control values (here $\mathcal{A} = [\alpha_{min}, \alpha_{max}] \equiv [-30°, 60°]$), $f : \mathbb{R}_+ \times \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}^d$ is the dynamics of the system as it appears in (2.2), $\mathcal{C} \subset \mathbb{R}^d$ is the target set to be reached in minimal time, and $\Psi : \mathbb{R}^d \times \mathcal{A} \to \mathbb{R}$ is the mixed state/control constraint function (here $\Psi(x, \alpha) := Q(r, v)|\alpha| - C_s$ with $Q(r, v) > 0$ for all (r, v)). Furthermore we introduce a state constraint in the form $g(y(t)) \leq 0$ (for some Lipschitz continuous function $g : \mathbb{R}^d \to \mathbb{R}$) in order to add additional constraints on the admissible set of trajectories. This will be also useful for numerical purposes.

The function $\mathcal{T}$ represents the minimal time needed to reach the target starting from any $x \in \mathbb{R}^d$ with an admissible trajectory obeying the state/control constraints. Since the final time t_f appears in the dynamics, the above setting of the control problem *is not standard*.

Time optimal control problems for time-*in*dependent dynamics have been studied in several papers. It is known that in this context the minimum time function satisfies the DPP and is characterized as the solution of a steady HJB equation [2]. Bokanowski et al. proved in [4] that the minimum time function associated to the time-dependent dynamics does not satisfy the DPP and cannot be characterized by means of a HJB equation. However, following [6], the function $\mathcal{T}$ is related to the determination of the "backward reachable sets" of the control system. More precisely, we define a *reachability function* $\vartheta(t, x)$ which is ≤ 0 if there exist an admissible trajectory $y(t)$ starting from x and reaching the target $\mathcal{C}$ before time t:

$$\vartheta(t, x) := \min_{\alpha} \varphi(y_x^\alpha(t)) \bigvee \max_{\theta \in [0,t]} g(y_x^\alpha(\theta)),$$

where $y = y_x^\alpha$ denotes the solution of $\dot{y}(s) = f(y(s), \alpha(s))$ on $[0, t]$ with $y(0) = x$, φ is a function s.t. $\varphi(x) = 1$ if $x \in \mathcal{C}$ and $\varphi(x) = 0$ elsewhere, and the controls $\alpha : (0, t_f) \to [\alpha_{min}, \alpha_{max}]$ are subject to the constraint $\Psi(y_x^\alpha(t), \alpha(t)) \leq 0$. Then, following [6], one can verify that

$$\mathcal{T}(x) = \min\{t \geq 0,\ \vartheta(t,x) \leq 0\}. \tag{2.3}$$

It can be proven under classical assumptions that ϑ is the unique lower semi-continuous solution (in the *viscosity sense*) of the HJB equation

$$\begin{cases} \min\left(\frac{\partial}{\partial t}\vartheta(t,x) + \max_{\alpha\in\mathcal{A},\Psi(x,\alpha)\leq 0}(-f(t,x,\alpha)\cdot\nabla\vartheta(t,x)),\ \vartheta(t,x)-g(x)\right) = 0 \\ \vartheta(0,x) = \varphi(x) \end{cases} \tag{2.4}$$

where $t > 0$, $x \in \mathbb{R}^d$, and ∇ denotes the gradient with respect to the x variable only.

Let us recall that the time t appearing in the solution $\vartheta(t,x)$ of the HJB equation (2.4) flows in the reversed direction with respect to the physical time. Indeed, the reachable sets $\Omega_t := \{x : \vartheta(t,x) \leq 0\}$ evolve from the target to the rest of the space, following the characteristics lines of the HJB equation, see [5] for details. Then, the function $\widetilde{m}(\cdot)$ (reversed evolution of the mass) can be computed together with $\vartheta(\cdot,x)$. In practical computation, at each time step $k\Delta t$, $k = 0,1,\ldots$, we compute the mass $\widetilde{m}(k\Delta t)$ and then we evaluate the dynamics (2.2).

Meanwhile the function ϑ is computed, we obtain the minimum time function $\mathcal{T}$ using (2.3). Afterwards, the minimal time function is used to recover the optimal control and optimal trajectories from any point of the state space. Since $\widetilde{m}$ depends only on time, and $\mathcal{T}$ is the minimal time needed to reach the target, it results that the minimal mass needed to reach the target from (r,v,γ) is given by $M_0^*(r,v,\gamma) := \widetilde{m}(\mathcal{T}(r,v,\gamma))$. In other words, the HJB approach is able to compute the minimal propellant mass needed to reach the target with a given payload, rather than to compute the maximal payload which can be carried on the target starting with a given propellant mass.

It is interesting to note the consequences of such a treatment of the mass variable. The optimal control reconstructed by means of the value function $\mathcal{T}$ will be in feedback form with respect to the state variables (r,v,γ) only, while it will be open-loop with respect to the mass m.

3. Numerical results. To solve the HJB equation (2.4) we use the combination of two techniques. The first one is the discretization of the equation by the Ultra Bee scheme (a finite difference type scheme), which has the nice property of computing the reachable sets with good accuracy without numerical diffusion (see e.g. [7]). The second one is an efficient data structure used to store and evaluate the function ϑ. At any time step, the function θ is actually computed only in a small region of the domain (*narrow band*). This region acts as an interface which separates the computed zone from the not-yet-computed zone, and it is updated at each time step [1]. Computation stops when the whole domain was covered by this moving interface. The *sparse semi-dynamic* data structure proposed in [5] allows us to store and retrieve efficiently the values of the nodes in the *narrow band*, speeding up the computation. The CPU time needed to compute the solution of the three-dimensional climbing problem is then comparable to that of a two-dimensional problem.

Furthermore, physical considerations allow us to restrict the computational domain, avoiding state regions never covered by the launcher (for example where both r is very small and v is very large). We first define the domain

$$U := [r_T + 450\,\mathrm{m},\ r_T + 450\times 10^3\,\mathrm{m}] \times [60\,\mathrm{ms}^{-1},\ 10500\,\mathrm{ms}^{-1}] \times [0,\ 90\,\mathrm{deg}],$$

where $r_T = 6378 \times 10^3\,\mathrm{m}$ is the Earth's mean radius. Then, we restrict the computation on the subdomain K of points $(r, v, \gamma) \in U$ such that $\ell_{min}(r) < v < \ell_{max}(r)$, with

$$\ell_{min}(r) := \max\left(\max((r-b_1)/a_1, (r-b_2)/a_2), \min(500, (r-b_3)/a_3)\right),$$
$$\ell_{max}(r) := (r-b_4)/a_4,$$

and

$$\begin{array}{llll} a_1 = 33.330, & b_1 = 6544700; & a_2 = 95.000, & b_2 = 6388000; \\ a_3 = 52.000, & b_3 = 6379000; & a_4 = 14.285, & b_4 = 6370900. \end{array}$$

This domain restriction allows us also to get a more favourable CFL condition when solving the HJB equation. In our case the CFL condition has the form

$$\max\left(\frac{\Delta t}{\Delta r}\|f_r\|_{L^\infty(K)}, \frac{\Delta t}{\Delta v}\|f_v\|_{L^\infty(K)}, \frac{\Delta t}{\Delta \gamma}\|f_\gamma\|_{L^\infty(K)}\right) \le 1$$

where $f = (f_r, f_v, f_\gamma)$ are the components of the dynamics given in (2.2) (see [5]).

The target $\mathcal{C}$ is assumed to be the GTO orbit. This orbit can be characterized by a set of two equations in variables (r, v, γ), hence it is a one-dimensional curve that can be numerically evaluated.

Once the function $\mathcal{T}$ is computed, we reconstruct the optimal feedback control law $\alpha^*(r, v, \gamma)$ using standard techniques described, for example, in [2](Appendix A). Then, we computed the optimal trajectory solving the system (2.1) by a 4th-order Runge-Kutta scheme. We will compare our results to a *reference trajectory*, which is a numerical solution obtained by the CNES by using a shooting method on the complete (6+1)-dimensional model.

The payload (initial condition for the time-reversed mass) is chosen as

$$M_{PL} = 15.37 \times 10^3\,\mathrm{kg}.$$

We denote by N_r, N_v and N_γ the number of nodes in the r-axis, v-axis, and γ-axis, respectively, and by N_α the number of nodes used to discretize $\mathcal{A}$.

Example 1. We approximate the optimal trajectory from the point $x_0 := (r_0, v_0, \gamma_0) = (r_T + 10.03\ \mathrm{km},\ 476.00\ \mathrm{ms}^{-1},\ 36.67\ \mathrm{deg})$, which belongs to the reference trajectory

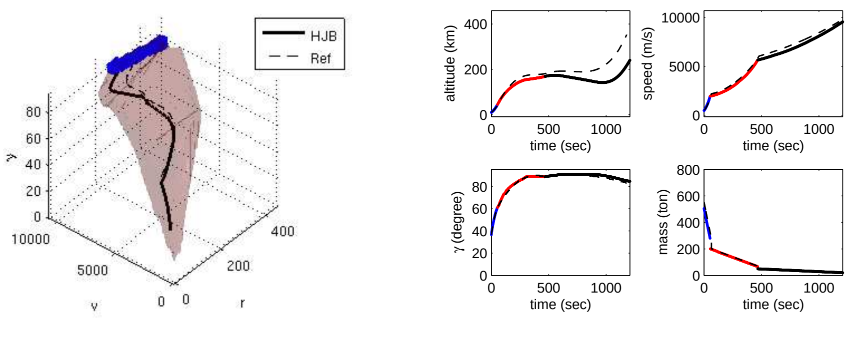

Fig. 3.1. *(Example 1) Left: optimal trajectory in the space (r, v, γ) (compared with the reference trajectory plotted in dotted line), and reachable set corresponding to $\mathcal{T}(r_0, v_0, \gamma_0)$. Right: $r(t)$, $v(t)$, $\gamma(t)$ and $m(t)$ along the optimal trajectory.*

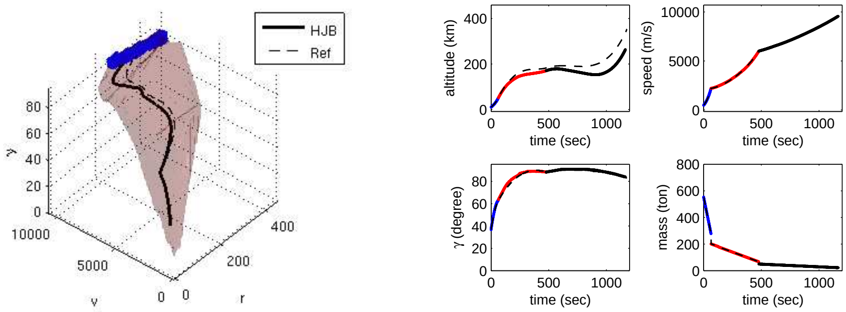

FIG. 3.2. *(Example 1) Trajectory starting with the initial mass* $M_{0,ref} = 553.87 \times 10^3$ *kg computed employing the optimal control* $\alpha^*(r, v, \gamma)$.

provided by the CNES. We solve (2.4) on a grid with $N_r = 100$, $N_v = 100$, $N_\gamma = 50$, and $N_\alpha = 30$. In Fig. 3.1, we have plotted the numerical solution obtained by the HJB approach, compared with the reference trajectory. We obtain the best point of the orbit to be reached with an optimal time $t_f = \mathcal{T}(x_0) = 1167.1\,\mathrm{s}$, (reference trajectories gives $1178.45\,\mathrm{s}$) and an optimal initial mass $M_0^* = 505.01 \times 10^3\,\mathrm{kg}$. The optimal trajectory gives a final mass of $13.74 \times 10^3\,\mathrm{kg}$, slightly below M_{PL} (the difference is due to the numerical errors and depends on the fact that the optimal control is not in feedback form with respect to the mass variable). The CPU time needed for the computation of the value function and the optimal trajectory was $253\,\mathrm{s}$.

We also tried to use the optimal control law obtained by the HJB approach to reconstruct a trajectory with the initial mass $M_{0,ref} = 553.87 \times 10^3\,\mathrm{kg}$, which is the mass provided by the CNES at x_0. We observe that this trajectory (Fig. 3.2) reaches the GTO target with a final mass of $15.89 \times 10^3\,\mathrm{kg}$, slightly above M_{PL}.

Example 2. Here we consider the optimal trajectory starting from the initial point $x_0 := (r_0, v_0, \gamma_0) = (r_T + 501.69\,\mathrm{m},\ 76.20\,\mathrm{ms}^{-1},\ 4.47\,\mathrm{deg})$. This point is taken from the reference trajectory and corresponds to the position of the launcher at the beginning of phase 1. In this case we are faced to an important numerical issue. The computation of the reachable set is very slow in the region close to the curve $(r, v) = (0, 0)$, due to a restrictive CFL condition. To fix this, we use new variables $(r, v, \gamma) \to (r', v', \gamma)$,

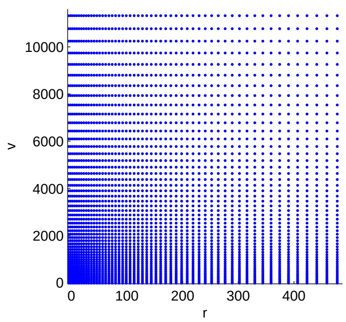

FIG. 3.3. *Mesh points in the* (r, v) *plane after the change of variable (3.1)*

defined implicitly by

$$r := K_r(e^{r'} - 1) + r_T, \quad v := K_v(e^{v'} - 1) + v_T, \tag{3.1}$$

with $K_r = 1.5 \times 10^3\,\mathrm{m}$, $K_v = 1165.6\,\mathrm{ms}^{-1}$, and $v_T = 10\,\mathrm{ms}^{-1}$, see Fig. 3.3. After this simple change of variables, we solve the HJB equation on a regular grid as before, in variables (r', v', γ).

Using a mesh size $N_r = 200$, $N_v = 200$, $N_\gamma = 75$, we obtain the optimal initial mass $M_0^* = 649.77 \times 10^3\,\mathrm{kg}$, corresponding to the optimal time $t_f = \mathcal{T}(x_0) = 1200.6\,\mathrm{s}$. The optimal trajectory gives a final mass of $13.27 \times 10^3\,\mathrm{kg}$. Results are shown in Fig. 3.4. CPU time was 57 min.

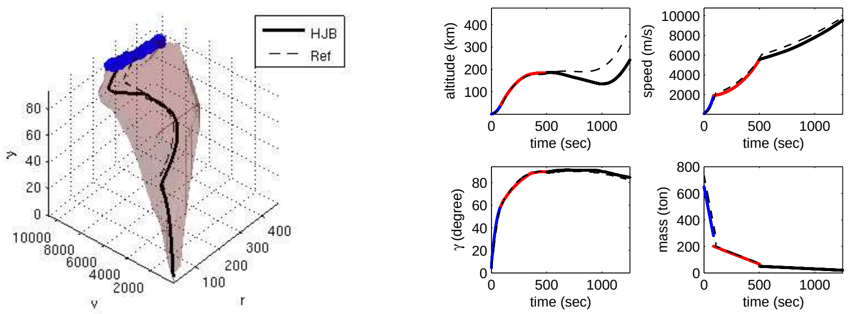

FIG. 3.4. *(Example 2) Left: optimal trajectory in the space (r, v, γ) (compared with the reference trajectory plotted in dotted line), and the reachable set corresponding to $\mathcal{T}(r_0, v_0, \gamma_0)$. Right: $r(t)$, $v(t)$, $\gamma(t)$ and $m(t)$ along the optimal trajectory.*

We also tried using the optimal control law obtained by the HJB method to compute a trajectory with the reference mass $M_{0,ref} = 736.39 \times 10^3\,\mathrm{kg}$, which is the mass of the launcher at x_0 as provided by the CNES. This trajectory reaches the GTO with a final mass of $16.07 \times 10^3\,\mathrm{kg}$, slightly above M_{PL}. Results are shown in Fig. 3.5.

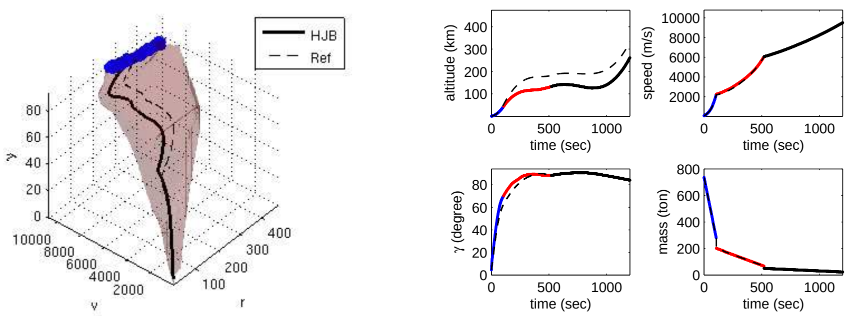

FIG. 3.5. *(Example 2) Trajectory starting with the initial mass $M_{0,ref} = 736.39 \times 10^3$ kg computed employing the optimal control $\alpha^*(r, v, \gamma)$.*

4. Conclusions. In this work we have studied the applicability of the HJB approach to a real-world climbing problem. In terms of accuracy, this approach is not yet competitive with the currently used methods. However, without requiring any *a* *priori* knowledge on the structure of the optimal trajectories, it can provide a qualitative global view of the backward reachable sets and give an approximation of an optimal feedback control and optimal trajectory. This information can be also used as initial guess for a more precise method (like, e.g., the shooting method), as proposed in [9]. Numerical results also suggest that the currently used launcher's trajectory could be actually improved. Indeed, tests have shown that given the payload M_{PL}, the launcher can reach the target starting with a lower amount of propellant. Alternatively, given the initial mass of propellant currently used, it is possible to reach the target with a heavier payload.

Further ongoing work concerns the study of the HJB approach including a ballistic phase, in order to reach the Geostationary orbit (GEO).

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