# Numerical Methods for Optimal Control Problems. Part II: Local Single-Pass Methods for Stationary HJ Equations

Ph.D. course in OPTIMAL CONTROL


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**(thanks to Simone Cacace for these slides!)**

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# Outline

- Hamilton-Jacobi equations for MTP
- Semi-Lagrangian discretization
- Classical iterative method
- Local single pass methods
- Fast marching method
- Can local single pass methods solve every HJ equation?

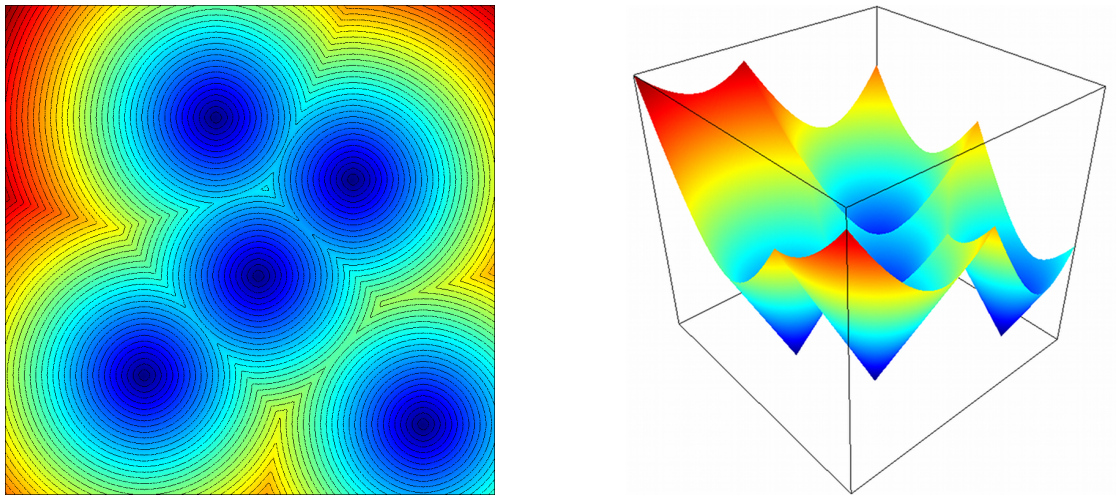
# Hamilton-Jacobi equations

Hamilton-Jacobi equations arise in several applied contexts, e.g. *front propagation, control problems and differential games.*

## Eikonal equation

$$\begin{cases} |\nabla v(x)| = 1 & x \in \mathbb{R}^d \setminus \mathcal{T} \\ v(x) = 0 & x \in \partial\mathcal{T} \end{cases}$$

The solution $v$ represents the *distance* function from $\partial\mathcal{T}$ and it is well understood in the framework of *viscosity solutions* [1].



Solution to the Eikonal equation in dimension $d = 2$ with $\mathcal{T} = \{\text{five random points}\}$.

[1] M.G. Crandall, P.-L. Lions, *Viscosity solutions of Hamilton-Jacobi equations*, Trans. Amer. Math. Soc., 277 (1983), 1–42.

# Semi-Lagrangian discretization of the HJB equation

Let $G$ be a structured grid with nodes $x_i$, $i = 1, \ldots, N$ and space step $\Delta x$.

## SL discretization of the HJB equation

$$w(x_i) = \min_{a \in A} \left\{ w(\tilde{x}_{i,a}) + \frac{|x_i - \tilde{x}_{i,a}|}{|f(x_i, a)|} \right\}, \qquad x_i \in G$$

where $\tilde{x}_{i,a}$ is a *non-mesh* point, obtained by integrating, until a certain final time $\hat{s}$, the ODE

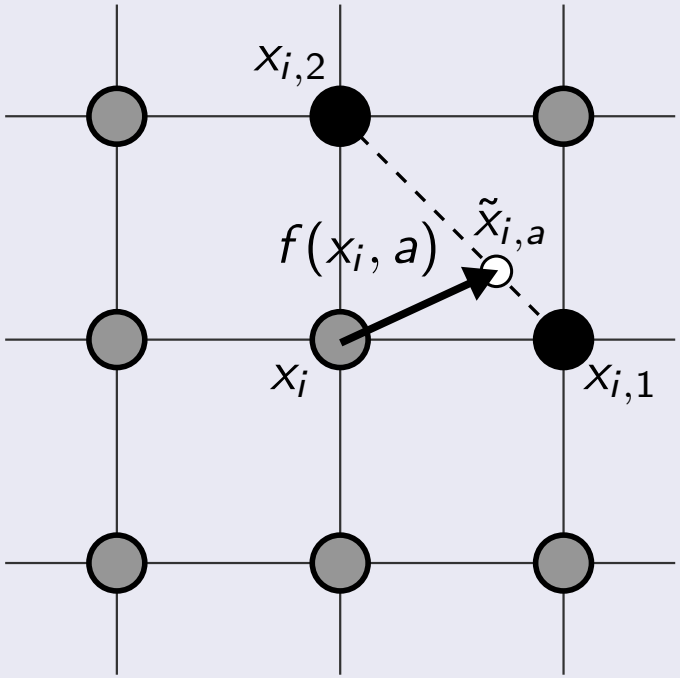
$$\begin{cases} \dot{y}(s) = f(y, a), & s \in [0, \hat{s}] \\ y(0) = x_i \end{cases}$$

and then setting $\tilde{x}_{i,a} = y(\hat{s})$. To make the scheme fully discrete, the set of admissible controls $A$ is discretized in $N_c$ points.

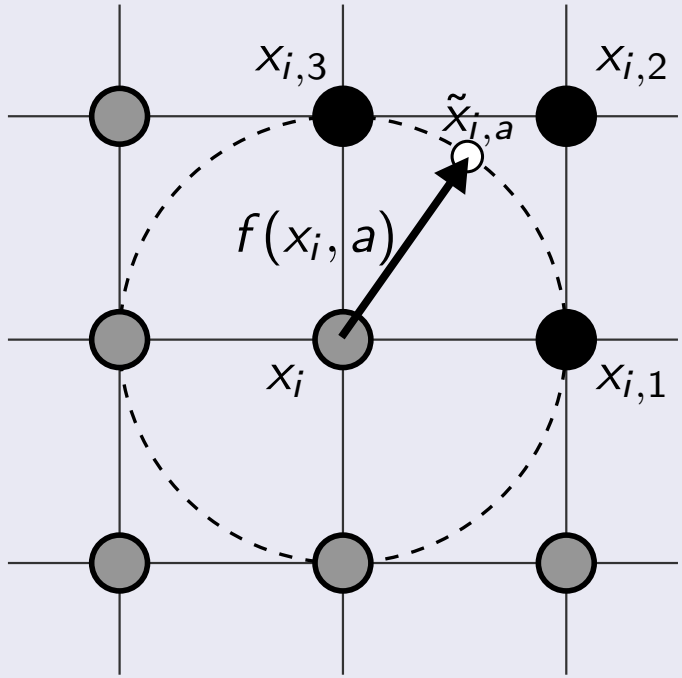
We get different versions of the SL scheme varying $\hat{s}$, the method used to solve the ODE and the interpolation method used to compute $w(\tilde{x}_{i,a})$.

# Semi-Lagrangian discretization of the HJB equation

## Explicit forward Euler scheme for the ODE + linear interpolation




2-points SL




3-points SL

2pSL: $\tilde{x}_{i,a}$ intercepts the line connecting $x_{i,1}$ and $x_{i,2}$.

3pSL: $\tilde{x}_{i,a}$ is at distance $\Delta x$ from $x_i$.

# Stationary Hamilton-Jacobi equations

Equations we are interested in can be recast as *minimum time problems*. By choosing the set of admissible controls $A = B_1(0)$ we get the following

## Reference equations

| $f(x,a)$ | HJ equation | Name |
|---|---|---|
| $a$ | $\lvert\nabla T(x)\rvert = 1$ | homogeneous eikonal |
| $c_1(x)a$ | $c_1(x)\lvert\nabla T(x)\rvert = 1$ | nonhomogeneous eikonal |
| $c_2(a)a$ | $c_2\left(\frac{\nabla T}{\lvert\nabla T\rvert}\right)\lvert\nabla T(x)\rvert = 1$ | hom. anisotropic eikonal |
| $c_3(x,a)a$ | $c_3\left(x, \frac{\nabla T}{\lvert\nabla T\rvert}\right)\lvert\nabla T(x)\rvert = 1$ | nonhom. anisotropic eikonal |

The functions $c_1$, $c_2$, $c_3$ are strictly positive and Lipschitz continuous.

# Classical iterative method

## How to solve the nonlinear system?

$$w(x_i) = S[w](x_i) := \min_{a \in A} \left\{ w(\tilde{x}_{i,a}) + \frac{|x_i - \tilde{x}_{i,a}|}{|f(x_i, a)|} \right\}, \qquad x_i \in G$$

## Fixed point algorithm

Given an initial guess $w^{(0)}$ iterate on the grid $G$

$$w^{(k)} = S[w^{(k-1)}] \qquad k = 1, 2, 3, \ldots$$

until

$$\max_{x_i \in G} |w^{(k)}(x_i) - w^{(k-1)}(x_i)| < \varepsilon$$

# Classical iterative method

## Pros

* Numerical approximation of *viscosity solution* in any dimension for any $f$.
* Easy implementation.
* Easy parallelization.
* *A priori* error estimates in $L^\infty$.
* Structured or unstructured grids.

## Cons

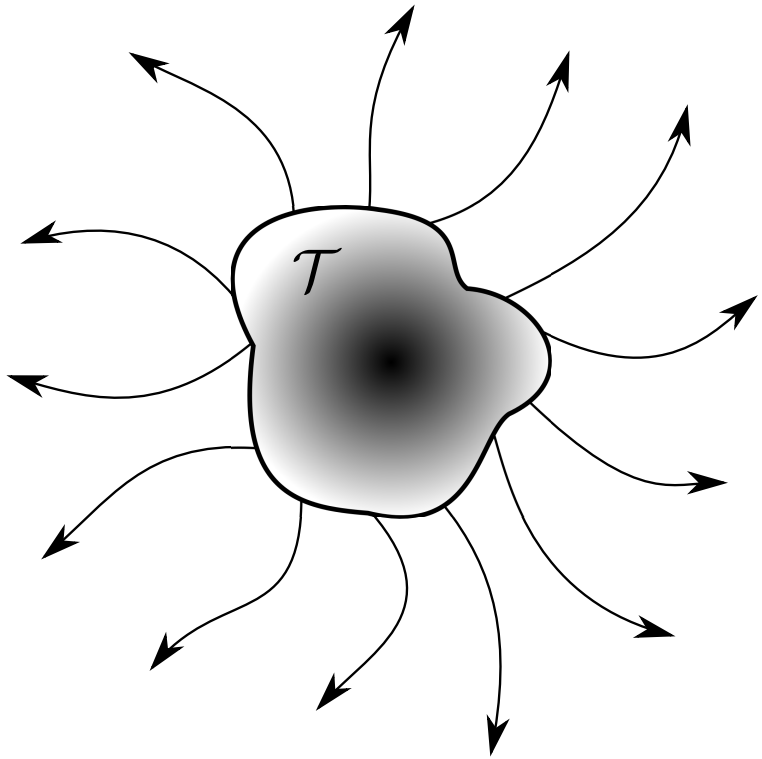
* "Curse of dimensionality" (exponentially increasingly nonlinear systems for high dimensional problems)

$\Rightarrow$ huge computational efforts

$\Rightarrow$ huge memory resources.

# Causality as source of efficiency

At the continuous level, information emanates from the target set $\mathcal{T}$ and propagates along characteristic lines.




By mimicking this behavior at the discrete level, one can produce a reordering of the grid nodes that decouples the nonlinear system.

# Local Single Pass algorithms

## Causality

Exploit physical/geometric properties of the HJ equations to find an ordering of the grid nodes that avoid useless computations.

## Locality

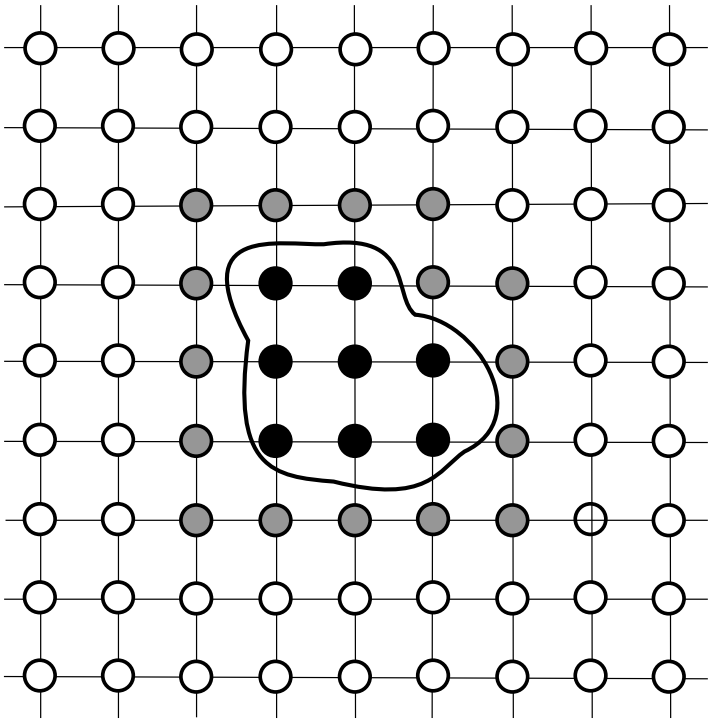
The computation is dynamically localized on the grid nodes carrying relevant information (few, compared to the entire grid).
Each node is computed using only neighboring nodes.

## Single Pass property

Each node is re-computed at most $r$ times, where $r$ only depends on the equation and the grid structure, not on the number of grid nodes.

# Fast Marching Method (FMM)

Inspired by Dijkstra's algorithm[3] for the *shortest path problem* on a graph, FMM (by Tsitsiklis[4] and Sethian[5]) is a *local single pass* method for the *Eikonal equation*.



●Accepted ●Considered ○Far

**FMM algorithm**

Set $T = 0$ in ACC and $T = +\infty$ in FAR
Compute $T$ in CONS

While(CONS $\neq \emptyset$)

  Find $\bar{x} = \arg\min_{x \in CONS} T(x)$

  Move $\bar{x}$ from CONS to ACC

  Move !ACC neighbors of $\bar{x}$ in CONS (if not yet in) and (re)compute $T$ on them

End While

[3] E. W. Dijkstra, *A note on two problems in connexion with graphs*, 1959.

[4] J. N. Tsitsiklis, *Efficient algorithms for globally optimal trajectories*, 1995.

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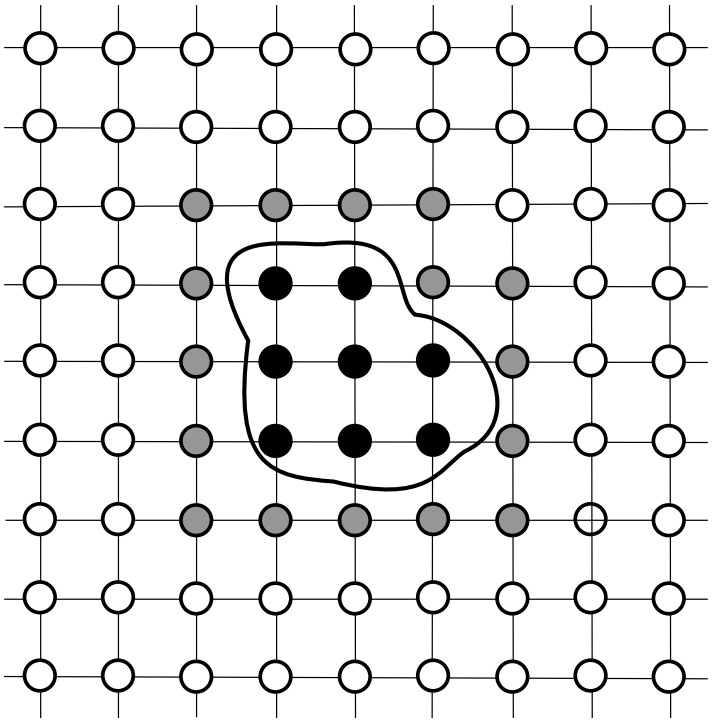

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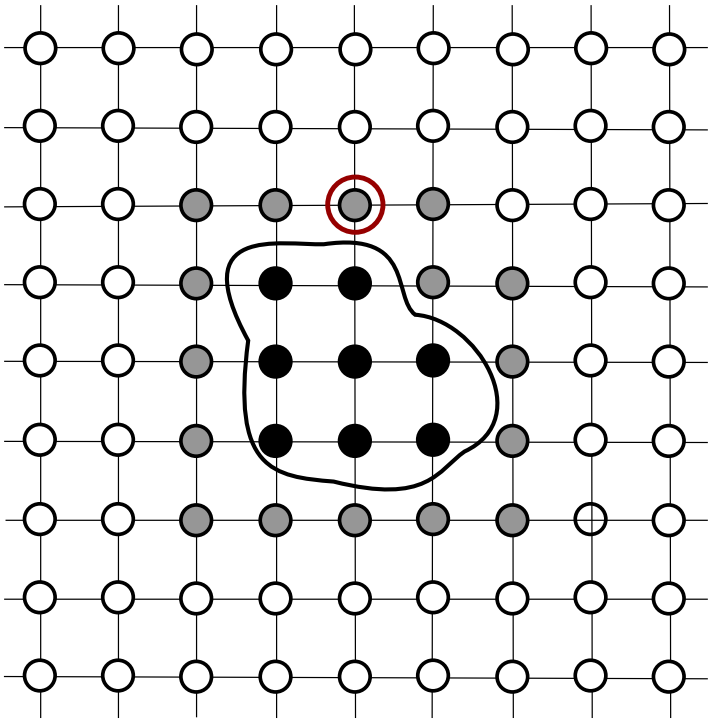

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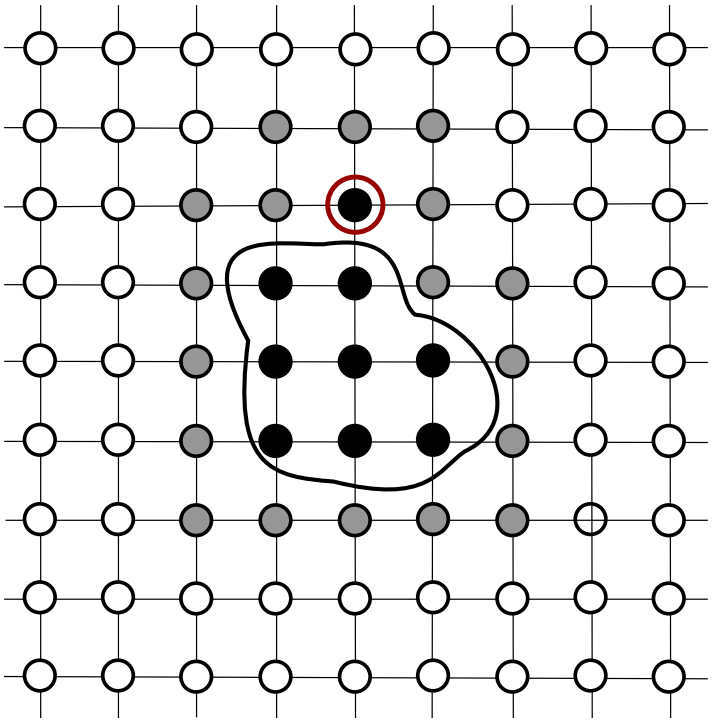

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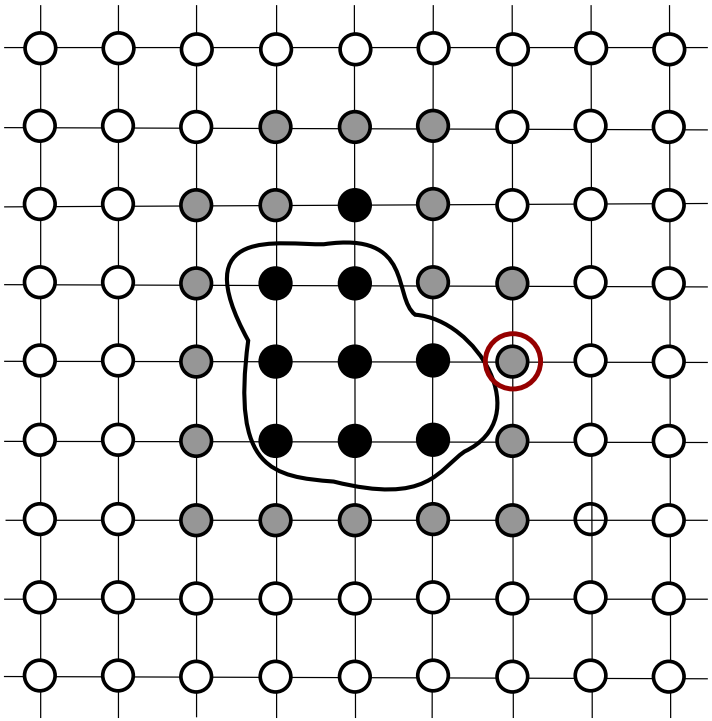

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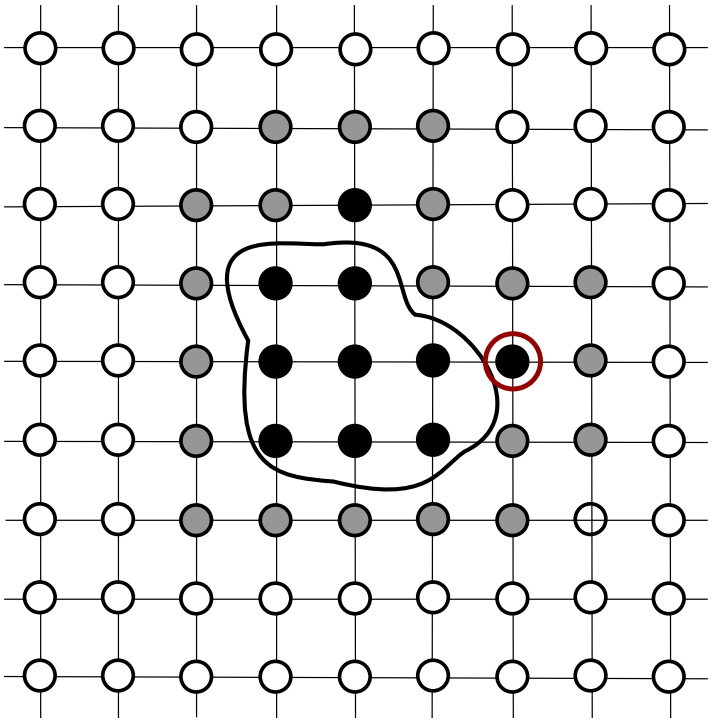

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# Why FMM works?

FMM computes each node in CONS by means of nodes with smaller values (practical implementations enforce the use of nodes in ACC only!).

The solution is computed in *ascending order*, so that the node in CONS with *minimal value* is the only *not influenced* by other nodes in CONS.

The *minimal value rule* corresponds to get information from the simplex containing $-\nabla T$ (and implies that CONS approximately expands as a *level set* of $T$).

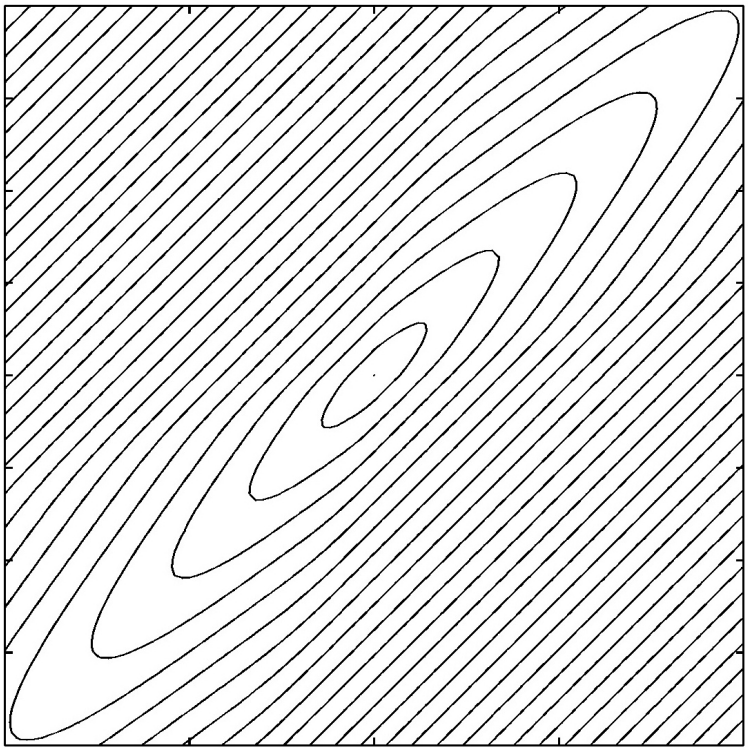
For the Eikonal equation, *characteristic lines coincide with gradient lines* of the solution itself, hence FMM computes the *correct* solution.

For general HJ equations this is *not true*!

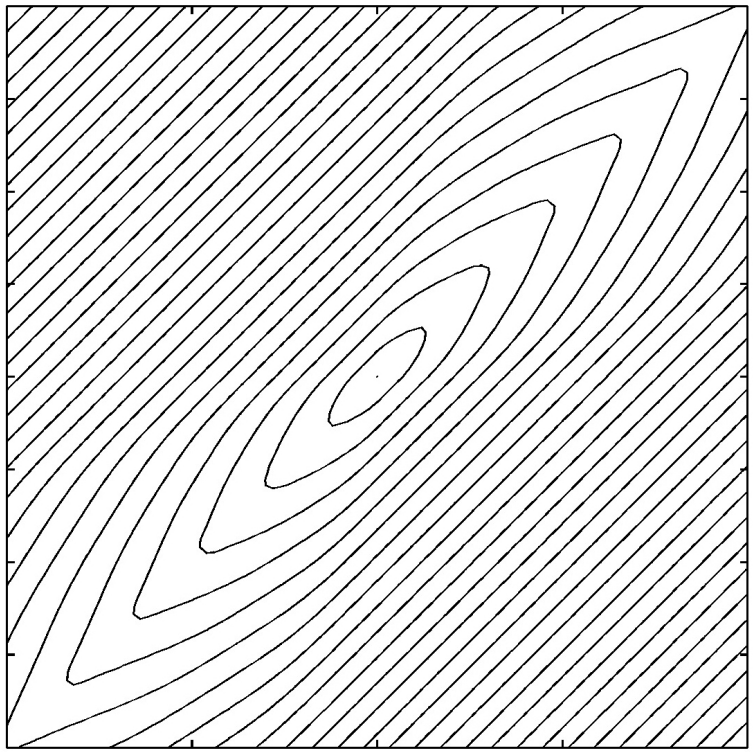
# FMM's FAILURE: Anisotropic Eikonal equation

$$f(x,a) = (1 + (\lambda\, a_1 + \mu\, a_2)^2)^{-\frac{1}{2}}\, a$$

$$a = (a_1, a_2) \in B_1(0), \quad \lambda, \mu > 0\,, \quad \mathcal{T} = (0,0)$$



EXACT



FMM

Wide divergence between characteristic and gradient lines!

# Beyond FMM

Several directions of research: high order accuracy, smart implementations, different schemes (FD, SL, DG, FV), other competitive approaches (FS, FI, MaxPlus), hybrid methods, more general HJ equations.

## Some references


K. Alton, I. M. Mitchell, *An ordered upwind method with precomputed stencil and monotone node acceptance for solving static convex Hamilton-Jacobi equations*, J. Sci. Comput., 51 (2012), pp. 313–348.

S. Cacace, E. Cristiani, M. Falcone, *Requiem for local sinlge-pass methods solving stationary Hamilton-Jacobi equations?*, submitted to SIAM J. Sci. Comput., preprint arXiv 1301.6775.

E. Carlini, M. Falcone, N. Forcadel, R. Monneau, *Convergence of a Generalized Fast Marching Method for an Eikonal equation with a velocity changing sign*, SIAM J. Numer. Anal., 46 (2008), pp. 2920–2952.

A. Chacon, A. Vladimirsky, *Fast two-scale methods for eikonal equations*, SIAM J. Sci. Comput., 34 (2012), pp. 547–578.

E. Cristiani, *A Fast Marching method for Hamilton-Jacobi equations modeling monotone front propagations*, J. Sci. Comput., 39 (2009), pp. 189–205.

E. Cristiani, M. Falcone, *Fast semi-Lagrangian schemes for the Eikonal equation and applications*, SIAM J. Numer. Anal., 45 (2007), pp. 1979–2011.

W.-K. Jeong, R. T. Whitaker, *A Fast Iterative Method for Eikonal Equations*, SIAM J. Sci. Comput., 30 (2008), pp. 2512–2534.

S. Kim, *An $O(N)$ level set method for eikonal equations*, SIAM J. Sci. Comput., 22 (2001), pp. 2178–2193.

W. M. McEneaney, *Max-Plus Methods for Nonlinear Control and Estimation*, Birkhauser Systems and Control Series, 2006.

J. A. Sethian, A. Vladimirsky, *Ordered upwind methods for static Hamilton-Jacobi equations: theory and algorithms*, SIAM J. Numer. Anal., 41 (2003), pp. 325–363.

H. Zhao, *A fast sweeping method for Eikonal equations*, Math. Comp., 74 (2005), pp. 603–627.

# Can Local Single Pass methods solve every HJ equation?

Let us classify HJ equations in two classes:

**(EIK)** Eikonal-like equations, whose characteristic lines coincide or lie in the same simplex of the gradient lines of their solutions.

**(¬EIK)** Non Eikonal-like equations, for which there exists at least a grid node where the characteristic line and the gradient of the solution do not lie in the same simplex.

By construction FMM works for equations of type EIK and fails for equations of type ¬EIK (e.g. the Anisotropic Eikonal equation).

**Is the minimal value rule really needed?**

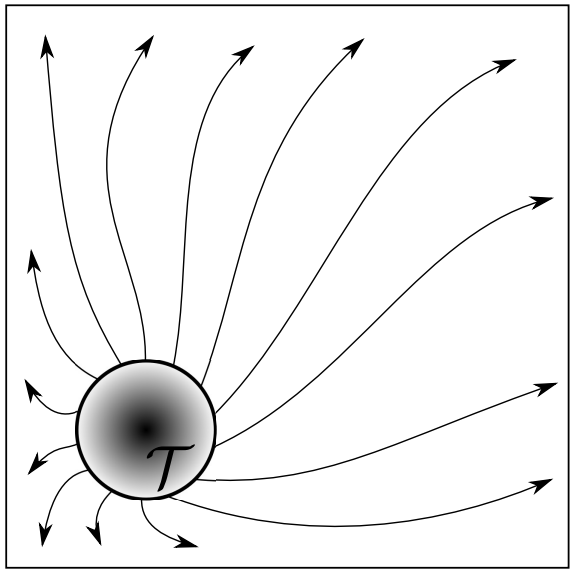
In order to solve ¬EIK equations, CONS *cannot be* at any time an approximation of a *level set*, i.e. we have to *drop* the minimal value rule.

# Can Local Single Pass methods solve every HJ equation?

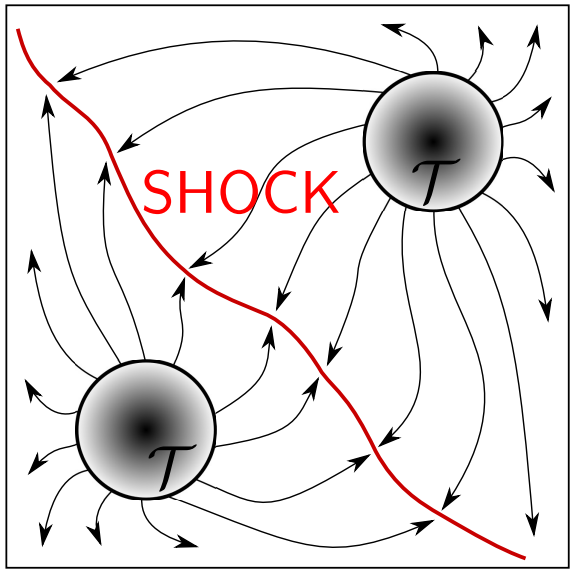
We consider another classification:

**(DIFF)** Equations with smooth characteristics. Information spreads from the target $\mathcal{T}$ to the rest of the space along smooth lines, without shocks. The solution $T$ is differentiable.

**(¬DIFF)** Equations with non smooth characteristics. Information starts from the target $\mathcal{T}$ and then crashes, creating shocks.
The solution $T$ is Lipschitz continuous.




DIFF



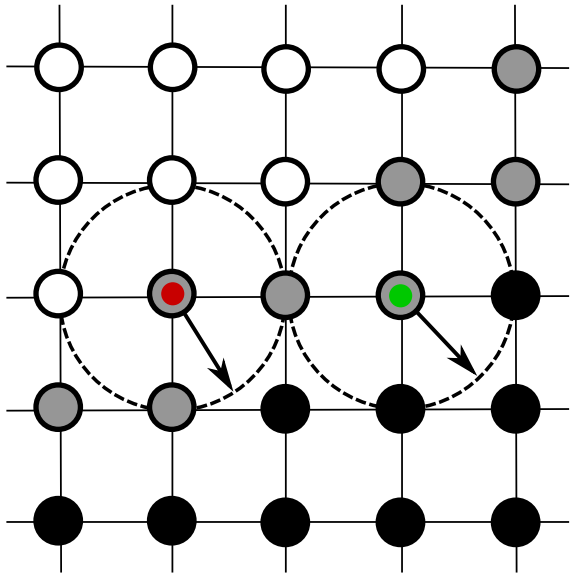

¬DIFF

# Can Local Single Pass methods solve every HJ equation?

$$T(x_i) = \min_{a \in A} \left\{ T(\tilde{x}_{i,a}) + \frac{|x_i - \tilde{x}_{i,a}|}{|f(x_i, a)|} \right\}, \qquad x_i \in G$$

## Safeness

A node $x_i \in$CONS is said to be *safe* if $T(x_i)$ is computed using values at neighboring interpolation points which are in ACC only.



Warning! Safeness makes sense if nodes in CONS can be computed using nodes both in ACC and CONS.

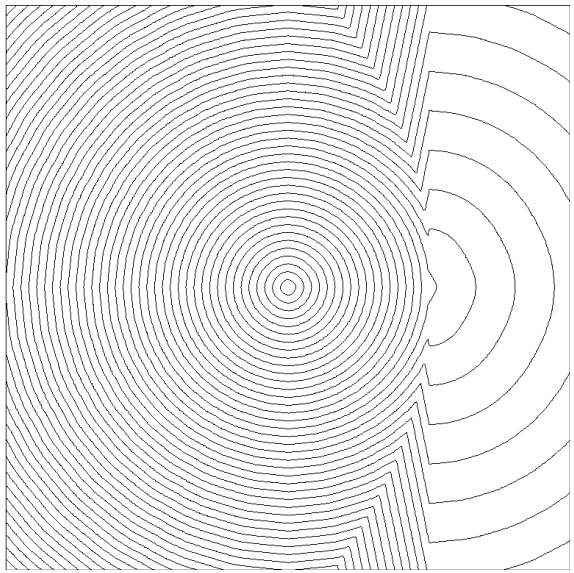
# Can Local Single Pass methods solve every HJ equation?

## Safe "Method" (SM)

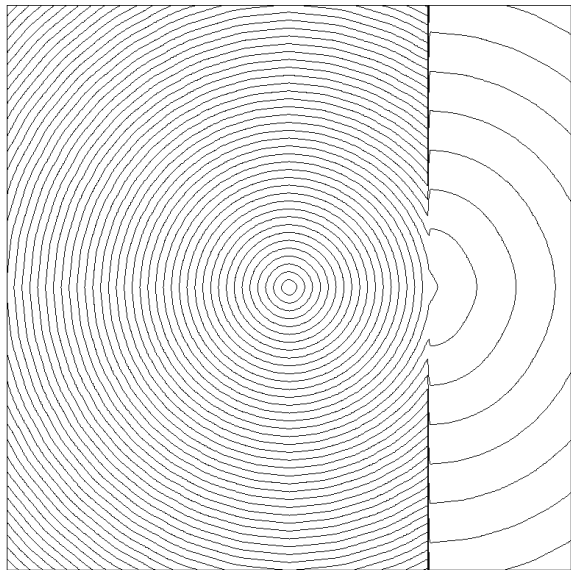
At each step, every *safe* node in CONS enters ACC.

SM *can* solve DIFF equations (both EIK and ¬EIK), it is much faster than FMM (multiple node acceptance, no search of min value in CONS).

SM fails for equations of type ¬DIFF.



EIK&¬DIFF: FMM works



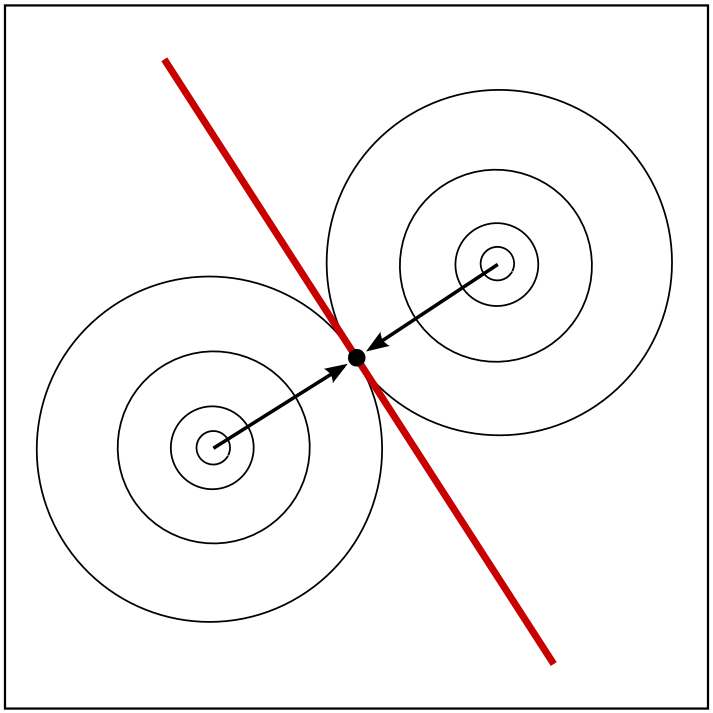
EIK&¬DIFF: SM fails

# Can Local Single Pass methods solve every HJ equation?

## How to handle the shocks?

As in the continuous case, a grid node $\Delta x$-close to a shock has to be approached by the ACC region approximately *at the same time* from the directions corresponding to the characteristic lines.

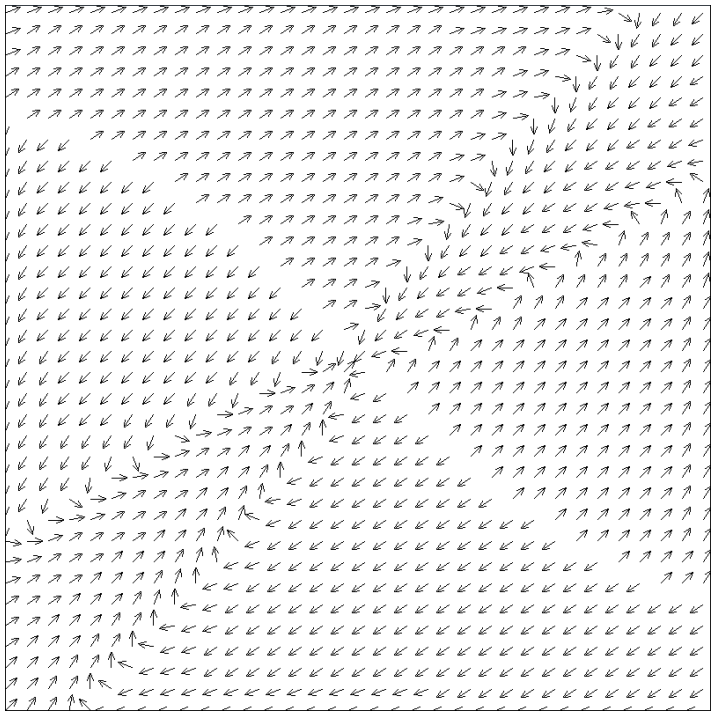
This property is satisfied by FMM in the case EIK, since CONS is approximately a *level set*.



# Requiem for Local Single Pass methods?

**¬EIK&¬DIFF equations are very hard (if not impossible) to solve**

¬EIK requires CONS not to be a level set, whereas
CONS $\sim$ level set seems the only way to handle shocks in ¬DIFF.



A shock crossing a region with strong anisotropy. What to do?


THANK
YOU