Numerical Methods for Optimal Control Problems. Part I: Hamilton-Jacobi-Bellman Equations and Pontryagin Minimum Principle

Ph.D. course in OPTIMAL CONTROL

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Main references

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Introduction

Controlled nonlinear dynamical system

$$\begin{cases} \dot{y}(s) = f(y(s), \alpha(s)), & s > t \\ y(t) = x \in \mathbb{R}^n \end{cases}$$

Solution:

$$y_{x,\alpha}(s)$$

Admissible controls: $\alpha \in \mathcal{A} := \{\alpha : [t, +\infty) \to A\}, \quad A \subset \mathbb{R}^m$

Regularity assumptions

Are they meaningful from the numerical point of view? Discussion.

Payoff

$$\max_{\alpha \in \mathcal{A}} J_{x,t}[\alpha]$$

Infinite horizon problem

$$J_{x,t}[\alpha] = \int_t^{\infty} r\Big(y_{x,\alpha}(s), \alpha(s)\Big) e^{-\mu s} ds, \quad \mu > 0$$

Finite horizon problem

$$J_{x,t}[\alpha] = \int_t^{T} r\Big(y_{x,\alpha}(s), \alpha(s)\Big) ds + g(y_{x,\alpha}(T))$$

Target problem

$$J_{x,t}[\alpha] = \int_t^{\tau} r\Big(y_{x,\alpha}(s), \alpha(s)\Big) ds, \quad \tau := \min\{s : y_{x,\alpha}(s) \in \mathcal{T}\}$$

HJB equation

Value function

$$v(x,t) := \max_{\alpha\in\mathcal{A}} J_{x,t}[\alpha], \qquad x\in\mathbb{R}^n,\ t\in[0,T]$$

Theorem (HJB equation)

Assume that $v\in C^1$. Then v solves

$$v_t(x,t)+\max_{a\in A}\{f(x,a)\cdot\nabla_x v(x,t)+r(x,a)\}=0, \quad x\in\mathbb{R}^n,\ t\in[0,T)$$

with the terminal condition

$$v(x,T)=g(x), \quad x\in\mathbb{R}^n$$

sup{J}=-inf{-J}

What about cost functionals to be minimized?

Find α^* by means of v

Given $v(x,t)$ for any $x \in \mathbb{R}^n$ and $t \in [0,T]$, we define

$$\alpha^*_{\text{feedback}}(x,t) := \arg\max_{a \in A}\{f(x,a) \cdot \nabla_x v(x,t) + r(x,a)\}$$

or, coming back to the original variables,

$$\alpha^*_{\text{feedback}}(y,s) := \arg\max_{a \in A}\{f(y,a) \cdot \nabla_x v(y,s) + r(y,a)\}.$$

Then, the optimal control is

$$\alpha^*(s) = \alpha^*_{\text{feedback}}(y^*(s), s)$$

where $y^*(s)$ is the solution of

$$\begin{cases} \dot{y}^*(s) = f(y^*(s), \alpha^*_{\text{feedback}}(y^*(s), s)), & s > t \\ y(t) = x \end{cases}$$

PMP

Theorem (Pontryagin Minimum Principle)

Assume α^* is optimal and y^* is the corresponding trajectory. Then there exists a function $p^* : [t, T] \to \mathbb{R}^n$ (costate) such that

$$\begin{cases} \dot{y}^*(s) = f(y^*(s), \alpha^*(s)) \\ \dot{p}^*(s) = -\nabla_x f(y^*(s), \alpha^*(s)) \cdot p^*(s) - \nabla_x r(y^*(s), \alpha^*(s)) \\ \alpha^*(s) = \arg\max_{a \in A} \left\{ f(y^*(s), a) \cdot p^*(s) + r(y^*(s), a) \right\} \end{cases}$$

with initial condition $y^*(t) = x$ and terminal condition $p^*(T) = \nabla g(y^*(T))$.

PMP can fail!

Along the optimal trajectory the Hamiltonian $H(y^*, p^*, a^*) = f^* \cdot p^* + r^*$ may not be an explicit function of the control inputs.

HJB↔PMP connection

Theorem

If $v \in C^2$, then

$$p^*(s) = \nabla_x v(y^*(s), s), \quad s \in [t, T].$$

The gradient of the value function gives the optimal value of the costate all along the optimal trajectory, in particular for $s = t$!

Direct methods

The control problem is entirely discretized and it is written in the form

Discrete problem

Find ξ^* such that $J(\xi^*) = \max_{\xi\in\mathbb{R}^n} J(\xi)$, with $J : \mathbb{R}^n \to \mathbb{R}$.

Fix a grid $(s^1, \ldots, s^n, \ldots, s^N)$ in $[t, T]$ with $s^n - s^{n-1} = \Delta s$. A discrete control function α is characterized by the vector $(\alpha^1, \ldots, \alpha^N)$ with $\alpha^n = \alpha(s^n)$.
Given $(\alpha^1, \ldots, \alpha^N)$, the ODE is discretized, for example, by

$$y^{n+1} = y^n + \Delta s\ f(y^n, \alpha^n).$$

and so it is the payoff, for example

$$J(\alpha) = \sum_n r(y^n, \alpha^n)\Delta s + g(y^N) + \text{penalization for ctrl and state constr.}$$

Then, a gradient method is used to maximize J.

Semi-Lagrangian discretization of HJB

$$v_t(x,t) + \max_{a\in A}\left\{f(x,a)\cdot\nabla_x v(x,t) + r(x,a)\right\} = 0, \quad x\in\mathbb{R}^n,\ t\in[0,T)$$

Fix a grid in $\Omega\times[0,T]$, with $\Omega\subset\mathbb{R}^n$ bounded. Steps: Δx, Δt. Nodes: $\{x_1,\ldots,x_M\}$, $\{t^1,\ldots,t^N\}$. Discrete solution: $w_i^n \approx v(x_i,s^n)$.

$$\frac{w_i^n - w_i^{n-1}}{\Delta t} + \max_{a\in A}\left\{\frac{w^n\big(x_i+\Delta t\ f(x_i,a)\big) - w_i^n}{\Delta t} + r(x_i,a)\right\} = 0$$

$$w_i^{n-1} = \max_{a\in A}\left\{\underbrace{w^n\big(x_i+\Delta t\ f(x_i,a)\big)}_{\text{to be interpolated}} + r(x_i,a)\right\} = 0$$

CFL condition (not needed but useful)

$$\Delta t\max_{x,a}|f(x,a)| \le \Delta x$$

Shooting method for PMP

Find the solution of $S(p_0) = 0$, where

$$S(p_0) := p(T) - \nabla g(y(T))$$

and $p(T)$ and $y(T)$ are computed solving the ODEs

$$\begin{cases} \alpha^n = \arg\max_{a \in A} \{f(y^n, a) \cdot p^n + r(y^n, a)\} \\ y^{n+1} = y^n + \Delta s \; f(y^n, \alpha^n) \\ p^{n+1} = p^n + \Delta s \; (-\nabla_x f(y^n, \alpha^n) \cdot p^n - \nabla_x r(y^n, \alpha^n)) \end{cases}$$

with only initial conditions

$$y^1 = x \quad \text{and} \quad p^1 = p_0$$

The solution of $S(p_0) = 0$ can be found by means of an iterative method like bisection, Newton, etc.

HJB↔PMP for numerics

Idea [CM10]

1. Solve HJB on a coarse grid
2. Compute $\nabla_x v(x,0) = p_0$
3. Use it as initial guess for the shooting method.

Advantages

1. Fast in dimension ≤ 4, feasible in dimension 5-6.
2. Highly accurate
3. Reasonable guarantee to converge to the global maximum of J.

HJB equation

Value function

$$v(x) := \min_{\alpha \in \mathcal{A}} J_x[\alpha], \quad x \in \mathbb{R}^n \quad (t = 0)$$

with cost functional

$$J_x[\alpha] = \int_0^\tau r\Big(y_{x,\alpha}(s), \alpha(s)\Big) ds + g(y_{x,\alpha}(\tau)), \quad \tau := \min\{s : y_{x,\alpha}(s) \in \mathcal{T}\}$$

Theorem (HJB equation)

Assume that $v \in C^1$. Then v solves the stationary equation

$$\max_{a \in A}\{-f(x,a) \cdot \nabla_x v(x) - r(x,a)\} = 0, \quad x \in \mathbb{R}^n \backslash \mathcal{T}$$

with boundary conditions

$$v(x) = g(x), \quad x \in \partial\mathcal{T}$$

HJB equation

Eikonal equation

If $f(x,a) = c(x)a$, $A = B(0,1)$, $r \equiv 1$, and $g \equiv 0$, we get

$$\max_{a \in B(0,1)} \{c(x)a \cdot \nabla v(x)\} = 1, \quad x \in \mathbb{R}^n \backslash \mathcal{T}$$

or, equivalently,

$$c(x)|\nabla v(x)| = 1, \quad x \in \mathbb{R}^n \backslash \mathcal{T}$$

with boundary conditions

$$v(x) = 0, \quad x \in \partial \mathcal{T}$$

Optimal trajectories $\equiv$ *characteristic lines* $\equiv$ *gradient lines*

PMP

Theorem (Pontryagin Minimum Principle)

Assume α^* is optimal and y^* is the corresponding trajectory. Then there exists a function $p^* : [0,\tau] \to \mathbb{R}^n$ (costate) such that

$$\begin{cases} \dot{y}^*(s) = f(y^*(s), \alpha^*(s)) \\ \dot{p}^*(s) = -\nabla_x f(y^*(s), \alpha^*(s)) \cdot p^*(s) - \nabla_x r(y^*(s), \alpha^*(s)) \\ \alpha^*(s) = \arg\max_{a \in A} \left\{ f(y^*(s), a) \cdot p^*(s) + r(y^*(s), a) \right\} \end{cases}$$

and

$$f(y^*(\tau), \alpha^*(s)) \cdot p^*(s) + r(y^*(s), \alpha^*(s)) = 0, \quad s \in [0, \tau] \quad \text{(HJB)}$$

with initial condition $y^*(0) = x$.

Semi-Lagrangian discretization of HJB

$$w_i = \min_{a\in A} \left\{ w\big(x_i + \Delta t\ f(x_i,a)\big) + \Delta t\ r(x_i,a) \right\}$$

Iterative solution

The fixed-point problem can be solved iterating the scheme until convergence, starting from any initial guess

$$w_i^{(k+1)} = \min_{a\in A} \left\{ w^{(k)}\big(x_i + \Delta t\ f(x_i,a)\big) + \Delta t\ r(x_i,a) \right\}$$

$$w_i^{(0)} = \begin{cases} +\infty & x_i \in \mathbb{R}^n \backslash \mathcal{T} \\ g(x_i) & x_i \in \partial\mathcal{T} \end{cases}$$

CFL condition (not needed but useful)

$$\Delta t \max_{x,a} |f(x,a)| \leq \Delta x$$

Shooting method for PMP

Find the solution of $S(p_0, \tau) = 0$ where

$$S(p_0, \tau) := \Big(y(\tau) - \mathcal{T}, f(y(\tau), \alpha(\tau)) \cdot p(\tau) + r(y(\tau), \alpha(\tau)) \Big)$$

and $y(\tau)$, $p(\tau)$ and $\alpha(\tau)$ are computed solving the ODEs

$$\begin{cases} \alpha^n = \arg\max_{a \in A} \{ f(y^n, a) \cdot p^n + r(y^n, a) \} \\ y^{n+1} = y^n + \Delta t \; f(y^n, \alpha^n) \\ p^{n+1} = p^n + \Delta t \; (-\nabla_x f(y^n, \alpha^n) \cdot p^n - \nabla_x r(y^n, \alpha^n)) \end{cases}$$

with only initial conditions

$$y^1 = x \quad \text{and} \quad p^1 = p_0$$